ON INTERNAL WAVES IN AN INHOMOGENEOUS FLUID

(O VNUTRENNYKH VOLNAKH V NEODNORODNOI ZHIDKOSTI)

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A.M. TER-KRIKOROV
(Moscow)

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It is known that the process of wave propagation in an inhomogeneous fluid differs essentially from the process of wave propagation in a homogeneous fluid.

This problem has been investigated in Lamb's book [1] and in [2] for very special density distributions.

Here, a detailed approximate investigation is carried out for arbitrary density and vorticity distribution. So-called long waves are studied in detail. The proposed asymptotic method can be used also to solve other boundary value problems of mathematical physics for rectilinear strips.

1. Formulation of the problem. Let us consider the steady flow of an ideal, heavy fluid with a free boundary over a smooth horizontal bottom. It is assumed that the fluid is incompressible but may be inhomogeneous. Let us select a coordinate system as in Fig. 1. Let $y = Y(x)$ be the equation of the free boundary, ρ the density of the fluid particles, p the hydrodynamic pressure, g the gravitational constant, $\mathbf{v}$ the velocity vector. If the vector $\mathbf{a} = \sqrt{\rho}\,\mathbf{v}$ is introduced, then in nondimensional variables we will have the equations of motion

$$\operatorname{div} \mathbf{a} = 0, \qquad \mathbf{a} \nabla \rho = 0 \tag{1.1}$$

$$(\mathbf{a} \nabla)\, \mathbf{a} = -\nu \rho \mathbf{y}^\circ - \nabla p \qquad \left(\nu = \frac{gH}{c^2}\right)$$

and the boundary conditions

$$a_n = 0 \quad \text{for } y = 0, \tag{1.2}$$

$$a_n = 0, \qquad p = \text{const} \quad \text{for } y = Y(x)$$

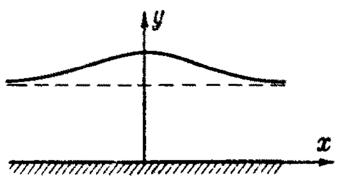

Fig. 1.

Here H is the fluid depth, c the characteristic velocity. The new measurement units are selected so that the fluid discharge, the flow vector **a** and the average depth will equal unity.

In addition to the boundary conditions (1.2), it is necessary to prescribe certain functions which will characterize the density distribution and the vorticity in the flow.

Let us note that the first of equations (1.1) permits the introduction of a stream function for the vector **a**

$$a_x = \partial\psi / \partial y, \qquad a_y = -\partial\psi / \partial x$$

Then it follows from the second equation (1.1) that the density is constant along a streamline, i.e. $\rho = \rho(\psi)$. The third of equations (1.1) can be written thus

$$\nabla h = \mathbf{a} \times \operatorname{rot} \mathbf{a} + \nu y \rho' (\psi) \nabla \psi \qquad \left(h = \frac{a^2}{2} + p + \nu\rho (\psi)\, y\right) \qquad (1.3)$$

Since the vector ∇h is orthogonal to the vector **a**, the function $h = h(\psi)$ depends only on ψ. The functions $\rho(\psi)$ and $h(\psi)$ should be considered assigned. The function $\rho(\psi)$ characterizes the density distribution among the streamlines and the function $h(\psi)$ the vorticity distribution. Now projecting equation (1.2) in the direction of the vector ∇_ψ, we obtain the equation

$$\triangle\psi = \nu\rho' (\psi)\, y - h' (\psi) \qquad (1.4)$$

Because of the first equation in (1.1), the function ψ is constant on the free boundary and on the bottom of the channel; moreover, because of the choice of the measurement units, the stream vector **a** equals unity across a transverse section of the flow, hence, we have the boundary conditions

$$\psi = 0 \quad \text{for } y = 0, \qquad \psi = 1, \ \frac{(\nabla\psi)^2}{2} + \nu\rho(1)\, Y (x) = \text{const} \quad \text{for } y = Y(x) \quad (1.5)$$

Hence, the problem is reduced to the determination of $Y(x)$ and $\psi(x,y)$ such that the function $\psi(x, y)$ will satisfy equation (1.4) and boundary conditions (1.5) in the strip $0 < y < Y(x)$.

It is inconvenient to investigate this form of the problem since the boundary conditions are assigned on an unknown boundary. This difficulty can be eliminated by a change of variable which makes ψ an independent variable.

2. Transformation of the equations. Let us first note that equation (1.4) is equivalent to the system of equations

$$\frac{\partial a_x}{\partial y} - \frac{\partial a_y}{\partial x} = -h'(\psi) + \nu\rho'(\psi)\, y, \qquad \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} = 0, \quad a_x = \frac{\partial \psi}{\partial y}, \quad a_y = -\frac{\partial \psi}{\partial x} \tag{2.1}$$

Let us take x and ψ as independent variables. Then the system (2.1) can be written thus:

$$a_x \frac{\partial a_y}{\partial \psi} - a_y \frac{\partial a_x}{\partial \psi} + \frac{\partial a_x}{\partial x} = 0, \quad a_x \frac{\partial a_x}{\partial \psi} + a_y \frac{\partial a_y}{\partial \psi} - \frac{\partial a_y}{\partial x} = -h'(\psi) + \nu\rho'(\psi)\, y \tag{2.2}$$

Moreover, as it is easy to see

$$\frac{\partial y}{\partial x} = \frac{a_y}{a_x}, \quad \frac{\partial y}{\partial \psi} = \frac{1}{a_x}, \qquad y = \int_0^{\psi} \frac{dt}{a_x(x, t)} \tag{2.3}$$

The boundary condition (1.5) takes the form

$$a_y = 0 \quad \text{for } \psi = 0, \quad a_x^2 + a_y^2 + 2\nu\rho(1)\, y(x, 1) = \text{const} \quad \text{for } \psi = 1 \tag{2.4}$$

Equations (2.2) to (2.3) with the boundary conditions (2.4) admit of a trivial solution corresponding to unperturbed plane-parallel fluid flow

$$a_y^\circ = 0, \qquad a_x^\circ = p(\psi), \qquad \frac{dp^2}{d\psi} = 2\left[-h'(\psi) + \nu\rho'(\psi)\int_0^{\psi} \frac{dt}{p(t)}\right]$$

If the function $h(\psi)$ is given, then $p(\psi)$ can be found from the last equation. However, it is more natural to prescribe the function $p(\psi) = \rho V^2$, twice the kinetic energy of the unperturbed flow per unit volume. Then the function $h(\psi)$ is expressed simply in terms of $p(\psi)$ and $\rho(\psi)$. Since the depth of the unperturbed flow must equal unity, then it follows from (2.3) that

$$\int_0^1 \frac{dt}{p(t)} = 1 \tag{2.5}$$

The boundary value problem for equations (2.2) is a typical nonlinear eigenvalue problem. The nonlinear theory should indicate the values of the parameter ν for which nontrivial solutions can be derived from the trivial solution. Let us interchange dependent and independent variables, assuming that $p > 0$

$$a_x = p(1 + u), \qquad a_y = pv, \qquad \eta = \int_0^{\psi} \frac{dt}{p(t)}$$

Because of (2.5), the strip $0 \leqslant \eta \leqslant 1$ corresponds to the strip $0 \leqslant \psi \leqslant 1$.

In the new variables, the boundary value problem for the system of equations (2.2) is reduced to the solution of the system of equations

$$\frac{\partial v}{\partial \eta} + \frac{\partial u}{\partial x} = v \frac{\partial u}{\partial \eta} - u \frac{\partial v}{\partial \eta} \tag{2.6}$$

$$\frac{\partial}{\partial \eta}[p^2(\eta) u] - p^2 \frac{\partial v}{\partial x} - \nu\rho'(\eta) \int_0^\eta u \, d\eta =$$

$$= -\frac{1}{2}\frac{\partial}{\partial \eta}[p^2(u^2+v^2)] - \nu\rho'(\eta)\int_0^\eta \frac{u^2}{1+u}\, d\eta$$

with the boundary conditions

$$v = 0 \qquad \text{for } \eta = 0 \tag{2.7}$$

$$\frac{\partial u}{\partial x} + \nu k v = -\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) + \nu k u v \frac{1}{1+u} \quad \text{for } \eta = 1 \qquad \left(k = \frac{\rho(1)}{p^2(1)}\right)$$

If this problem is solved, then the equation of the family of streamlines is given by the equality (2.3) in which it is only necessary to make the change of variables

$$y(x, \eta) = \int_0^\eta \frac{dt}{1+u(x,t)} \tag{2.8}$$

3. Small amplitude waves. By discarding the nonlinear terms in (2.6) and the boundary conditions (2.7) we obtain

$$\frac{\partial v}{\partial \eta} + \frac{\partial u}{\partial x} = 0, \qquad \frac{\partial}{\partial \eta}[p^2(\eta) u] - p^2(\eta)\frac{\partial v}{\partial x} - \nu\rho'(\eta)\int_0^\eta u\, d\eta = 0 \tag{3.1}$$

$$v = 0 \quad \text{for } \eta = 0, \qquad \frac{\partial u}{\partial x} + \nu k v = 0 \quad \text{for } \eta = 1$$

Furthermore, let us require that the functions v and u be bounded at infinity. Eliminating v we arrive at a linear boundary and eigenvalue problem for the function $v(x, \eta)$

$$\frac{\partial}{\partial \eta}\left[p^2(\eta)\frac{\partial v}{\partial \eta}\right] + p^2(\eta)\frac{\partial^2 v}{\partial x^2} - \nu\rho'(\eta)\, v = 0 \tag{3.2}$$

$$v = 0 \quad \text{when } \eta = 0, \qquad \frac{\partial v}{\partial \eta} - \nu k v = 0 \quad \text{when } \eta = 1$$

The eigenvalues of the boundary value problem (3.2) have the form

$$v_n(x, \eta) = \sin \omega (x - x_0)\, z_n(\eta) \tag{3.3}$$

where $z_n(\eta)$ is an eigenfunction of the Sturm-Liouville problem for the ordinary equation

$$Lz = \frac{d}{d\eta}\left[p^2 \frac{dz}{d\eta}\right] - \left[\omega^2 p^2 + \nu \frac{d\rho}{d\eta}\right] z = 0 \tag{3.4}$$

$$z(0)=0, \qquad \frac{dz}{d\eta}-\nu kz=0 \quad \text{when } \eta=1$$

In what follows, we shall everywhere assume that

$$p(\eta)>0, \qquad d\rho\,/\,d\eta<0 \tag{3.5}$$

Let us note that (3.4) is not the customary Sturm-Liouville problem since the parameter ν enters both the equation and the boundary conditions. However, the same method can be used to investigate such a boundary problem as is used for the Sturm-Liouville problem [3] by proving that under conditions (3.5) the eigenvalues and eigenfunctions have the properties expressed in the following lemmas.

Lemma 3.1. All the eigenvalues are simple and real.

Lemma 3.2. The eigen numbers form an infinite denumerable set ν_1, ν_2, $\ldots$, ν_m, $\ldots$. For large m the following asymptotic formulas are valid

$$z_m(\eta)=b(\eta)\sin\left[\sqrt{\nu_m}\,\zeta(\eta)\right]+O(m^{-1}), \qquad \nu_m=\frac{m^2\pi^2}{\zeta^2(1)}+O(1) \tag{3.6}$$

$$\zeta(\eta)=\int_0^\eta f(t)\,dt, \qquad b^{-2}(\eta)=p^2(\eta)\,f(\eta), \qquad f(\eta)=\frac{1}{p(\eta)}\left(-\frac{d\rho}{d\eta}\right)^{\frac{1}{2}}$$

Lemma 3.3. The inhomogeneous boundary value problem

$$Lu=\Phi(\eta), \qquad u(0)=0, \qquad \left[\frac{du}{d\eta}-\nu ku\right]_{\eta=1}=F(\nu)$$

is solvable if ν is not an eigenvalue. If $\nu=\nu_m$ is an eigenvalue then for the inhomogeneous problem to be solvable it is necessary that the condition

$$\int_0^1 z_m(\eta)\,\Phi(\eta)\,d\eta-z_m(\eta)\,p^2(1)\,F(\nu)=0 \tag{3.7}$$

be satisfied.

Let us give the proof of Lemma 3.1 as an example. Since the differential operator L is of second order then the eigenvalues of the boundary value problem (3.4) are simple. In order to prove that they are real, let us note that, for any two eigenfunctions, the relation

$$kp^2(1)\,u(1)\,v(1)-\int_0^1\frac{d\rho}{d\eta}\,u(\eta)\,v(\eta)\,d\eta=0 \tag{3.8}$$

can be satisfied, which is easily deduced if the expression $vLu-uLv$ is integrated between 0 and 1 and the boundary conditions are used. If $u(\eta)$ is an eigenfunction corresponding to the complex eigenvalue $\nu = \alpha + i\beta$, then $\mu = \alpha - i\beta$ will also be an eigenvalue to which the eigenfunction $u(\eta)$ will correspond.

Because of (3.8)

$$kp^2(1)\,|u(1)|^2 - \int_0^1 \frac{d\rho}{d\eta}\,|u(\eta)|^2\,d\eta = 0$$

But this equality is impossible since, by assumption, $p(\eta) > 0$, $d\rho/d\eta < 0$, $k > 0$. Hence, Lemma 3.1 is proved.

Lemma 3.3 can be proved just as simply if the Green's function $G(\eta, \eta', \nu)$ is constructed for the operator L. The solvability condition is derived in the usual manner if a Laurent series expansion of the Green's function in the neighborhood of the eigenvalue $\nu = \nu_m$ is used.

Substituting the expression (3.3) for $v(x, \eta)$ in the system of equations (3.1), we find $u(x, \eta)$ and then substituting $u(x, \eta)$ into (2.8) and linearizing the obtained expression, we find the equation of the family of streamlines

$$u\,(x,\eta) = A\cos\omega(x - x_0)\frac{dz_m(\eta)}{d\eta}, \qquad v\,(x,\eta) = A\omega\sin\omega(x - x_0)\,z_m(\eta)$$

$$y\,(x,\eta) = \eta - A\cos\omega(x - x_0)\,z_m(\eta)$$

Here A is an arbitrary parameter (the amplitude).

The deviation of the streamlines in the unperturbed flow from the corresponding streamlines in the plane-parallel flow is characterized by the quantity $z_m(\eta)$. It follows from the asymptotic formulas for $z_m(\eta)$ presented in the Lemma 3.2 that for large m the quantity $z_m(1)$ is of the order of m^{-1} so that the free boundary is slightly perturbed. The function $\zeta(\eta)$ grows monotonically from 0 to $\zeta(1)$ when η grows from 0 to 1. Hence, such numbers η_k^m exist that

$$\zeta(\eta_k{}^m) = \frac{2k+1}{2m}\,\zeta(1), \qquad \text{if} \quad 2k+1 < 2m$$

For large m the extrema of the function $z_m(\eta)$ will be reached on the streamlines $\eta = \eta_k{}^m$. It can be said that wave channels for waves with number m exist at the depths $y = \eta_k{}^m$. Hence, for waves with large numbers the maximum waviness will be at a certain depth rather than at the free surface.

From (1.1) we obtain an expression for the propagation velocity

$$c_m = \sqrt{gh\nu_m^{-1}} \tag{3.9}$$

Here ν_m is an eigen number of the boundary value problem (3.4). The eigenvalues ν_m depend on the wavelength.

Therefore, in an inhomogeneous fluid there exists a denumerable set of waves of prescribed length, where the wave with the number m is propagated with velocity c_m. This is the essential difference from the case of a homogeneous fluid where a single wave of a prescribed length exists.

Let us note that solutions of the problem have been sought which are bounded at infinity. The problem of looking for solutions having a prescribed periodicity could have been formulated. As is known, in the case of a homogeneous fluid there exists a denumerable set of waves possessing a prescribed periodicity. Their lengths are obtained by dividing the length of the fundamental wave into an integer number of parts. In the case of an inhomogeneous fluid, rather than the one fundamental wave there exists a denumerable set of fundamental waves which correspond to the eigen numbers $\nu = \nu_m$. The lengths of the remaining waves are also obtained by dividing the lengths of the fundamental waves into an integer number of parts. In the sequel a nonlinear theory of so-called long waves, i.e. waves whose length is large compared with the depth of the fluid, is constructed.

As in the case of the flow of a homogeneous fluid, linear theory cannot describe certain physical phenomena which occur with long waves. Thus, within the scope of linear theory, it is impossible to construct a solitary wave which is obtained from a periodic wave as a result of a passage to the limit, when the wavelength approaches infinity.

If λ is the wavelength then the quantity $1/\lambda$ is a natural small parameter. However, it is impossible to look for the solution in the form of a power series in $1/\lambda$ since the coefficients of an expansion of a periodic function in powers of $1/\lambda$ will not be periodic functions. But if a preliminary extension of the horizontal independent variable is made [1] then such an expansion is possible.

4. Let us expand equations (2.6) and conditions (2.7) in the small parameter ε and let us look for u, v and ν in the form of power series in ε

$$\xi = \varepsilon x, \qquad u = \varepsilon^2 u_1 + \varepsilon^4 u_2 + \ldots, \qquad v = \varepsilon^3 v_1 + \varepsilon^5 v_2 + \ldots,$$
$$\nu = \nu_0 (1 + \varepsilon^2 \nu_1 + \ldots) \tag{4.1}$$

The physical meaning of the parameter ε will be clarified later. Let us write the equations and boundary conditions for the first and second approximations

$$\frac{\partial v_1}{\partial \eta} + \frac{\partial u_1}{\partial \xi} = 0, \qquad \frac{\partial}{\partial \eta}\,[p^2 u_1] - \nu_0 \rho'\,(\eta) \int_0^\eta u_1 d\eta = 0 \tag{4.2}$$

$$v_1 = 0 \quad \text{for } \eta = 0, \qquad \frac{\partial u_1}{\partial \xi} + \nu_0 k v_1 = 0 \quad \text{for } \eta = 1$$

$$\frac{\partial v_2}{\partial \eta} + \frac{\partial u_2}{\partial \xi} = v_1 \frac{\partial u_1}{\partial \eta} - u_1 \frac{\partial v_1}{\partial \eta} \tag{4.3}$$

$$\frac{\partial}{\partial \eta}\,[p^2\,(\eta)\, u_2] - \nu_0 \rho'\,(\eta) \int_0^\eta u_2 d\eta =$$

$$= \nu_0 \nu_1 \rho'\,(\eta) \int_0^\eta u_1 d\eta - \frac{1}{2}\frac{\partial}{\partial \eta}\,(p^2 u_1^2) - \nu_0 \rho'\,(\eta) \int_0^\eta u_1^2 d\eta + p^2 \frac{\partial v_1}{\partial \xi}$$

$$v_2 = 0 \text{ for } \eta = 0, \quad \frac{\partial u_2}{\partial \xi} + \nu_0 k v_2 = - k \nu_0 \nu_1 v_1 - \frac{1}{2}\frac{\partial u_1^2}{\partial \xi} + \nu_0 k u_1 v_1 \text{ for } \eta = 1$$

Let us put

$$v_1 = - C'\,(\xi)\, w\,(\eta), \qquad u_1 = C\,(\xi)\, w'\,(\eta) \tag{4.4}$$

It is impossible to determine the function $C(\xi)$ from equation (4.2) but to determine $w(\eta)$ it is necessary to solve the boundary value problem for the ordinary differential equation

$$Lw = \frac{d}{d\eta}\left[p^2\,(\eta)\frac{dw}{d\eta}\right] - \nu_0 \rho'\,(\eta)\, w = 0, \quad w\,(0) = 0$$
$$\left[\frac{dw}{d\eta} - k\nu_0 w\right]_{\eta=1} = 0 \tag{4.5}$$

This is a particular case of problem (3.4) already investigated in Section 3. From the reasoning presented there it follows that nontrivial solutions of this problem exist for $\rho'(\eta) < 0$ when the parameter ν_0 takes on one of the values $\nu_0^{(1)}$, $\nu_0^{(2)}$, ... to which the eigenfunctions w_1, w_2, ... correspond. In what follows let us assume that ν_0 is one of the eigen numbers and $w(\eta)$ the corresponding eigenfunction. The function $C(\xi)$ must be determined from the second approximation.

Let us turn to the investigation of the second approximation. Substituting the expression (4.4) for v_1 and u_1 into the first of equations (4.3), we obtain

$$\frac{\partial v_2}{\partial \eta} + \frac{\partial u_2}{\partial \xi} = CC'\left[2w'^2 - \frac{d}{d\eta}\,(ww')\right] \tag{4.6}$$

Let us put

$$u_2 = \frac{\partial\omega}{\partial\eta}, \qquad v_2 = -\frac{\partial\omega}{\partial\xi} + CC'\left[2\int_0^\eta w'^2(t)\,dt - ww'\right] \tag{4.7}$$

Substituting these expressions into the second equation of (4.3) and in the boundary conditions, we obtain

$$L\left(\frac{\partial\omega}{\partial\xi}\right) = \Phi, \quad \omega = 0 \quad \text{for } \eta = 0, \quad \frac{\partial}{\partial\eta}\left(\frac{\partial\omega}{\partial\xi}\right) - \nu_0 k\frac{\partial\omega}{\partial\xi} = F \quad \text{for } \eta = 1 \tag{4.8}$$

where

$$\Phi = \nu_0\nu_1\rho' C'w - 2CC'\left[\frac{1}{2}\frac{d}{d\eta}(p^2w'^2) + \nu_0\rho'\int_0^\eta w'^2 d\eta\right] - p^2wC'''$$

$$F = \nu_0\nu_1 kC'w(1) + CC'\left[w'^2(1) + 2\nu_0 k\int_0^1 w'^2(\eta)\,d\eta\right] \tag{4.9}$$

The inhomogeneous boundary value problem was used to determine $\partial\omega/\partial\xi$ and since ν_0 is an eigen number, then for this boundary value problem to be solvable the equality (3.7) must be satisfied. In the case under consideration, (3.7) is written as follows:

$$\int_0^1 w(\eta)\,\Phi(\xi, \eta)\,d\eta - w(1)\,p^2(1)\,F = 0 \tag{4.10}$$

Since the functions Φ and F depend on the function $C(\xi)$ and its derivatives, then the equality (4.10) is a differential equation for $C(\xi)$

$$\gamma C''' - \alpha CC' - \nu_1\beta C' = 0 \tag{4.11}$$

where

$$\alpha = -\int_0^1 w(\eta)\left\{[p^2w'^2]' + 2\nu_0\rho'\int_0^\eta w'^2 dt\right\}d\eta +$$

$$+ w(1)\,p^2(1)\left[w'^2(1) + 2\nu_0 k\int_0^1 w'^2(\eta)\,d\eta\right]$$

$$\beta = -\int_0^1 \nu_0\rho' w^2(\eta)\,d\eta + \nu_0 kw'(1)\,p^2(1), \qquad \gamma = \int_0^2 p^2(\eta)\,w^2(\eta)\,d\eta$$

The expressions for the coefficients α and β can be simplified if it is recalled that the function $w(\eta)$ satisfies the differential equation

$$(p^2w')' = \nu_0 w d\rho / d\eta$$

By integrating by parts, it is easily established that the sum of all the non-integral components will be zero because of the boundary conditions so that we finally obtain

$$\alpha = 3\int_0^1 p^2(\eta)\, w'^3(\eta)\, d\eta, \qquad \beta = \int_0^1 p^2(\eta)\, w'^2(\eta)\, d\eta \tag{4.12}$$

Let $\alpha \neq 0$. Let us make the change of function

$$C = 9\gamma\zeta / \alpha + \nu_1\beta / \alpha$$

in equation (4.11). Then it will reduce to the known differential equation

$$\zeta''' = 9\zeta\zeta'$$

The solution of this equation is [5]

$$\zeta = \frac{a^2}{3}\left\{2k^2 - 1 - 3k^2 cn^2\left[\frac{a\sqrt{3}}{2}(\xi - \xi_0)\right]\right.$$

so that

$$C(\xi) = \frac{3a^2\gamma}{\alpha}\left\{2k^2 - 1 - 3k^2 cn^2\left[\frac{a\sqrt{3}}{2}(\xi - \xi_0)\right]\right\} + \nu_1\frac{\beta}{\alpha} \tag{4.13}$$

It is seen from (4.13) that the wave is symmetric relative to the vertical $\xi = \xi_0$ and if this axis of symmetry is taken as the vertical coordinate axis, the $\xi_0 = 0$. Let us note also that the arbitrary constant a in the final formula will enter only in the combination εa and, without limiting the generality, it is possible to take $a = 2/\sqrt{3}$ since only the relation of ε to the physical parameters of the problem can be changed hereby.

Under these simplifications we obtain for $C(\xi)$

$$C(\xi) = \frac{4\gamma}{\alpha}[2k^2 - 1 - 3k^2 cn^2 \xi] + \nu_1\frac{\beta}{\alpha}$$

Here $C(\xi)$ is a function with period $2K(k)$, where $K(k)$ is the complete elliptic integral of the second kind. The equalities (4.1) and (4.4) now yield

$$u = \varepsilon^2 w'(\eta)\, C(\varepsilon x) + O(\varepsilon^4), \qquad v = -\varepsilon^3 C'(\varepsilon x)\, w(\eta) + O(\varepsilon^5)$$

Now substituting the expression for u into the equality (2.8) also, we obtain the equation for the family of streamlines

$$y = \eta - \varepsilon^2 w(\eta)\, C(\varepsilon x) + O(\varepsilon^4)$$

The solution depends on ε, k and ν_1. But ν_1 can be expressed in terms of ε and k, using the condition that the average depth of the flux is equal to unity

$$\nu_1 = -\frac{1}{\beta}\left\{2k^2 - 1 - \frac{3k^2}{K(k)}\int_{-K(k)}^{K(k)} cn^2(t)\,dt\right\} \tag{4.14}$$

Evidently the wavelength equals $\lambda = 2K(k)/\varepsilon$. Formula (3.9) yields an expression for the propagation velocity

$$c^2 = \frac{gh}{\nu_0}[1 - \varepsilon^2\nu_1(k)] + O(\varepsilon^4) \tag{4.15}$$

As has already been noted above, the parameter ν_0 can take on a denumerable set of values $\nu_0^{(1)}$, $\nu_0^{(2)}$, ..., $\nu_0^{(m)}$, ... A two-parameter family of streamlines corresponds to each value of ν_0. If the asymptotic formulas (3.6) are used, it is easy to establish that for fixed ε and k the larger the number of the wave, the smaller will its amplitude and propagation velocity be.

A solitary wave is obtained as a particular case for $k = 1$ when the wavelength becomes infinite. Substituting $k = 1$ in the appropriate formulas we find

$$\nu_1 = -\frac{4}{\beta}, \quad y = \eta + \frac{12\gamma}{\alpha}\varepsilon^2 w(\eta)\,\mathrm{sech}^2\,\varepsilon x, \quad c^2 = \frac{gh}{\nu_0}\left[1 + \frac{4\varepsilon^2\gamma}{\beta}\right] \tag{4.16}$$

Since $\beta > 0$, the propagation velocity will always be greater than the critical velocity equal to $\sqrt{(gh)/\nu_0}$. In contrast to a homogeneous fluid depression waves can be propagated in an inhomogeneous fluid.

It was assumed that the coefficient $\alpha \neq 0$. If $\alpha = 0$, then the proposed asymptotic process must be modified somewhat. In the general case it is necessary to make an expansion of the form $\xi = \varepsilon^k x$ and to look for the solution in the form of the series (4.1). If k is taken greater than 2, then since $\alpha = 0$ the boundary value problem obtained in the second approximation will be solvable and it is necessary to rely upon the equation of the third (kth in the general case) approximation to determine the arbitrary functions. The calculations are complicated.

5. Some examples. Various examples of the flow of an inhomogeneous fluid can be obtained by assigning the functions $\rho(\eta)$ and $p(\eta)$ which characterize the density and vorticity distributions in the flow, in various manners. In many important cases the following assumptions can be made on the character of these functions

$$p(\eta) = 1 + O(\delta), \qquad \rho = 1 + O(\delta), \qquad \frac{d\rho}{d\eta} = -\delta + O(\delta)$$

where δ is small. If higher order infinitesimals in δ are discarded in (4.5), then it is easy to obtain

$$\frac{d^2w}{d\eta^2} + \nu_0\delta w = 0, \qquad w(0) = 0, \qquad \left[\frac{dw}{d\eta} - \nu_0 w\right]_{\eta=1} = 0 \tag{5.1}$$

Here the relation $k = \rho(1)/p^2(1) = 1 + O(\delta)$ has been used. The eigenfunctions of the boundary problem (5.1) have the form

$$w_k(\eta) = \sin \varkappa_k \eta \qquad (k = 0, 1, 2, \ldots)$$

where κ_k are the roots of the transcendental equations

$$\tan \varkappa = \frac{\delta}{\varkappa}, \qquad \varkappa^2 = \nu_0\delta$$

Let us note that for small δ the following approximate formulas are correct

$$\varkappa_0 = \sqrt{\delta} + O(\delta), \qquad \varkappa_n = n\pi + O(\delta), \qquad (n = 1, 2, \ldots)$$

Let us evaluate the coefficients γ, α and β by means of formulas (4.12). For κ_0 to higher order accuracy

$$\gamma_0 = \frac{\delta}{3}, \qquad \beta_0 = \delta, \qquad \alpha_0 = 3\delta^{\frac{3}{2}}$$

Substituting the expressions obtained into (4.16), we obtain for the solitary wave

$$y = \eta + 4/3\, \varepsilon^2\eta \operatorname{sech}^2 \varepsilon x, \qquad c^2 = gh(1 + 4/3\varepsilon^2)$$

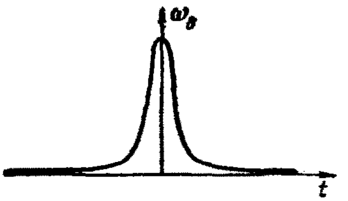

Fig. 2.

If $4\varepsilon^2/3$ is denoted by a (amplitude), then the Boussinesq-Rayleigh formulas are obtained

$$y = \eta\,[1 + a \operatorname{sech}^2 (3a/4)^{1/2} x], \quad c^2 = gh(1 + a)$$

Hence, for small δ the solitary wave, which is similar to the corresponding solitary wave for a homogeneous fluid, corresponds to the eigen number κ_0.

Let us now consider the case $\kappa = \kappa_n$. In this case, (4.12) yield

$$\gamma = 1/2, \quad \beta = 1/2\, n^2\pi^2, \quad \alpha = O(\delta)$$

We obtain for the family of streamlines and the propagation velocity

$$y = \eta + \frac{6}{\alpha}\varepsilon^2 \sin(n\pi\eta) \operatorname{sech}^2 \varepsilon x, \qquad c^2 = \frac{gh\delta}{n^2\pi^2}\left(1 + \frac{4\varepsilon^2}{n^2\pi^2}\right)$$

Evidently the propagation velocities of these waves are small for small δ.

Also of interest is the case when the density varies abruptly in the neighborhood of certain streamlines, i.e. when

$$\frac{d\rho}{d\eta} = \sum_{k=1}^{m} \alpha_k \omega_\delta^{(k)} (\eta - \eta_k) = -\Omega_\delta(\eta)$$

where $\omega_\delta^{(k)}(t)$ is a δ-function. For example, a function of the form $\omega_\delta(t) = \delta/\pi(t^2 + \delta^2)$, shown in Fig. 2. As $\delta \to 0$ these functions degenerate into the Dirac delta-function, which corresponds to the model of an m-layer fluid. In this case, in order to determine the function $\omega(\eta)$ it is necessary to solve the boundary value problem

$$\frac{d^2w}{d\eta^2} + \nu_0\Omega_\delta(\eta)\, w = 0, \qquad w(0) = 0, \qquad \left[\frac{dw}{d\eta} - \nu_0 k w\right]_{\eta=1} = 0 \tag{5.2}$$

This problem can be reduced by the customary method to the integral equation

$$w = \nu_0 \int_0^1 G(\eta, \eta', \nu_0)\, w(\eta')\, \Omega_\delta(\eta')\, d\eta'$$

where $G(\eta, \eta', \nu_0)$ is the Green's function of the operator $d^2/d\eta^2$ with the boundary conditions (5.2)

$$G(\eta, \eta', \nu_0) = \eta\left[\frac{\nu_0}{\nu_0 - 1}\eta' - 1\right] \quad (\eta \leqslant \eta'), \qquad G(\eta, \eta', \nu_0) = \eta'\left[\frac{\nu_0}{\nu_0 - 1}\eta - 1\right] \quad (\eta \geqslant \eta')$$

If δ is small, then by using the properties of the δ-function, we obtain

$$w(\eta) = -\nu_0 \sum_{k=1}^{m} \alpha_k w(\eta_k)\, G(\eta, \eta_k, \nu_0)$$

In order to find the eigen numbers, let us put $y = \eta_j$

$$w(\eta_j) = -\nu_0 \sum_{k=1}^{m} \alpha_k w(\eta_k)\, G(\eta_j, \eta_k, \nu_0), \qquad (j = 1, 2, \ldots m)$$

Equating the determinant of this system to zero, we obtain an equation for ν_0. Then finding the numbers $w(\eta_k)$ and substituting them into (5.3) we obtain an expression for the eigenfunctions. Let us examine the simplest case of a two-layer fluid, then

$$w(\eta) = -\nu_0\alpha_1 G(\eta, \eta_0, \nu_0)$$

Let us consider the jump in the density to be small and $w(\eta_0)$ to equal unity. To determine ν_0 we then obtain the equation

$$-\alpha_1\nu_0 G(\eta_0, \eta_0, \nu_0) = 1, \qquad -\alpha_1\nu_0\eta_0\left[\frac{\nu_0}{\nu_0 - 1}\eta_0 - 1\right] = 1$$

This quadratic equation has two roots, one of which approaches one as $\alpha_1 \to 0$ and the other approaches infinity. To the accuracy of higher order quantities in α_1 we obtain

$$\nu_0^{(1)} = 1 - \alpha_1 \eta_0^2, \qquad \nu_0^{(2)} = \frac{1}{\alpha_1 \eta_0 (1 - \eta_0)}$$

The corresponding eigenfunctions have the form

$$w_1(\eta) = \eta + O(\alpha), \qquad w_2(\eta) = \begin{cases} \eta(1-\eta_0) & (\eta \leqslant \eta_0) \\ \eta_0(1-\eta) & (\eta \geqslant \eta_0) \end{cases}$$

The function $w_1(\eta)$ achieves a maximum at $\eta = 1$ and $w_2(\eta)$ at $\eta = \eta_0$. Let us now find the solitary waves corresponding to $\nu_0^{(1)}$ and $\nu_0^{(2)}$. The solitary wave for $\nu_0^{(1)}$ will be close to the corresponding solitary wave in a homogeneous fluid.

Let us examine the wave corresponding to $\nu_0^{(2)}$. By means of (4.12) we obtain

$$\gamma = \frac{1}{3} \eta_0^2 (1 - \eta_0)^2, \qquad \beta = \eta_0 (1 - \eta_0), \qquad \alpha = 3\eta_0 (1 - \eta_0)(1 - 2\eta_0)$$

Substituting these expressions into (4.16), we obtain

$$y = \eta + \frac{4}{3} \frac{\eta_0 (1 - \eta_0)}{1 - 2\eta_0} \varepsilon^2 w_2(\eta) \operatorname{sech}^2 \varepsilon x, \qquad c^2 = gh\alpha_1 \eta_0 (1 - \eta_0) \left[1 + \frac{4}{3} \eta_0 (1 - \eta_0)\right]$$

Let us note that the propagation velocity of this wave is small, waviness is lacking on the upper boundary and the maximum waviness is achieved on the interface of the two layers. If $\eta_0 < 1/2$ then the wave has a solitary hump and if $\eta_0 > 1/2$ then it has a solitary trough. For $\eta_0 = 1/2$ the coefficient $\alpha = 0$ and the series in the small parameter must be constructed by means of the scheme proposed at the end of Section 4.

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